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The Atiyah–Hitchin bracket and the open Toda lattice☆

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Abstract

The dynamics of the finite nonperiodic Toda lattice is an isospectral deformation of the finite three-diagonal Jacobi matrix. It is known since the work of Stieltjes that such matrices are in one-to-one correspondence with their Weyl functions. These are rational functions mapping the upper half-plane into itself. We consider representations of the Weyl functions as a quotient of two polynomials and exponential representation. We establish a connection between these representations and recently developed algebraic-geometrical approach to the inverse problem for Jacobi matrix. The space of rational functions has natural Poisson structure discovered by Atiyah and Hitchin. We show that an invariance of the AH structure under linear-fractional transformations leads to two systems of canonical coordinates and two families of commuting Hamiltonians. We establish a relation of one of these systems with Jacobi elliptic coordinates.

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1. Introduction

The Toda lattice is a mechanical system of N particles connected by elastic strings. The Hamiltonian of the system is

$$H=\sum_{k=0}^{N-1}\frac{p_k^2}{2}+\sum_{k=0}^{N-2}\mathrm{e}^{q_k-q_{k+1}}.$$

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Introducing the classical Poisson bracket

$$\{f, g\} = \sum_{k=0}^{N-1} \frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k},$$

we write the equations of motion as

$$q_k^{\bullet} = \{q_k, H\} = p_k, \qquad p_k^{\bullet} = \{p_k, H\} = -\mathrm{e}^{q_k - q_{k+1}} + \mathrm{e}^{q_{k-1}-q_k}, \quad k=1, \dots, N-1.$$

We put $q_{-1} = -\infty, q_N = \infty$ in all formulae. Following [10,19] introduce the new variables

$$c_k = \mathrm{e}^{q_k - q_{k+1}/2}, \qquad v_k = -p_k.$$

In these variables

$$H = \sum_{k=0}^{N-1} \frac{v_k^2}{2} + \sum_{k=0}^{N-2} c_k^2,$$

and

$$\{c_k, v_k\} = -\frac{c_k}{2}, \qquad \{c_k, v_{k+1}\} = \frac{c_k}{2}. \tag{1.1}$$

The equations of motion take the form

$$v_k^{\bullet} = \{v_k, H\} = c_k^2 - c_{k-1}^2, \qquad c_k^{\bullet} = \{c_k, H\} = \frac{c_k(v_{k+1} - v_k)}{2}.$$

These equations are compatibility conditions for the Lax equation $L^{\bullet} = [A, L]$, where

$$L = \begin{bmatrix} v_0 & c_0 & 0 & \cdots & 0 \\ c_0 & v_1 & c_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & c_{N-3} & v_{N-2} & c_{N-2} \\ 0 & \cdots & 0 & c_{N-2} & v_{N-1} \end{bmatrix},$$

and

$$2A = \begin{bmatrix} 0 & c_0 & 0 & \cdots & 0 \\ -c_0 & 0 & c_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -c_{N-3} & 0 & c_{N-2} \\ 0 & \cdots & 0 & -c_{N-2} & 0 \end{bmatrix}.$$

The Lax formula implies that the spectrum $\lambda_0 < \cdots < \lambda_{N-1}$ of L remain fixed. It is known since the work of Stieltjes [25], that the rational function $w(\lambda) = (R(\lambda)\delta(0), \delta(0))$, where $R(\lambda) = (L - \lambda I)^{-1}$ is the resolvent, plays the key role in reconstruction of the matrix L

from its spectral data. It was encountered later in the spectral theory of the Sturm–Liouwille operator [28], and received the name of Weyl function. Simply by expanding it in the continued fraction

$$w(\lambda) = -\cfrac{1}{\lambda - v_0 - \cfrac{c_0^2}{\lambda - v_1 - \cfrac{c_1^2}{\lambda - v_2 - \cdots \cfrac{c_{N-2}^2}{\lambda - v_{N=1}}}}},$$

one can read the entries of L. This fact is central to Moser's solution of the nonperiodic Toda [21]. At the same time Stieltjes method appears to be a computational trick and does not provide a conceptual explanation why the function $w(\lambda)$ actually determines L.

An attempt to solve an inverse spectral problem for the finite Jacobi matrix using algebraic-geometrical approach was made at the beginning of 1980s by McKean [20]. It was realized that the corresponding spectral curve is a singular reducible Riemann surface.

The recent interest in the spectral curves of finite Toda lattice stems from various sources. One is the work of Seiberg and Witten [23,24] on supersymmetric Yang–Mills theories. The smooth hyperelliptic spectral curves of the periodic Toda appear in the pure gauge $N = 2$ SUSY Yang–Mills models in four dimensions [12]. The physically interesting limit of the theory which corresponds to transition from the smooth hyperelliptic curve to the singular one of McKean [20] was considered in [5]. Another source of interest is the Camassa–Holm equation [7]. It is known, see [3,22], that the dynamics of the so-called peakons solutions is isomorphic to an isospectral flow on the space of finite Jacobi matrices.

A solution of nonperiodic Toda within algebraic-geometrical approach was obtained recently in [18]. The poles of $w(\lambda)$ determine the curve and the zeros specify the divisor of the Baker–Akhiezer function. This information uniquely specifies the BA function and therefore the matrix L. Whence that fact that $w(\lambda)$ determines the matrix can be considered as a consequence of the Riemann–Roch theorem which guarantees uniqueness of the BA function. Using BA functions we obtained in [18] the explicit formulae for the solution, the symplectic structure and two systems of Darboux coordinates for it.

The present paper combines ideas developed in [27] with algebraic-geometrical approach of Krichever and Vaninsky [18]. It can be divided into two parts. In the first part of the paper (Sections 2–5) we establish relations between the standard objects of spectral theory and algebraic-geometrical constructions. We show how the BA function can be constructed from the orthogonal polynomial of the first kind and suitably normalized Weyl solution. The rational functions which map the upper half-plane into itself are defined by the formula

$$w(\lambda) = \frac{\rho_0}{\lambda_0 - \lambda} + \cdots + \frac{\rho_{N-1}}{\lambda_{N-1} - \lambda}, \qquad \rho_k > 0. \tag{1.2}$$

They are parameterized by $2N$ parameters ρ's and λ's. The rational functions corresponding to the Weyl functions of finite Jacobi matrices are specified by the condition $\sum \rho_k = 1$.

We consider different representations of the Weyl functions. The first one, the ratio of two monic polynomials,

$$w(\lambda) = -\frac{q(\lambda)}{p(\lambda)}$$

leads to the standard Abel map. Another, the exponential representation, which employs Krein's spectral shift function $\xi(z)$

$$w(\lambda) = \frac{1}{\lambda_0 - \lambda}\exp(\Xi(\lambda)), \quad \Xi(\lambda) = \int \frac{\xi(z)}{z-\lambda}\,\mathrm{d}z$$

produces the Abel map in Baker's form.

In the second part (Sections 6–9), we study the Atiyah–Hitchin bracket on rational functions and its relation to the Hamiltonian formalism for the Toda lattice. As it was recently discovered by Faybusovich and Gehtman [9], the AH bracket on rational functions can be written as

$$\{w(\lambda), w(\mu)\} = \frac{(w(\lambda)-w(\mu))^2}{\lambda-\mu}. \tag{1.3}$$

The bracket (1.1) corresponds to the restriction of (1.3) on the $(2N-1)$-dimensional submanifold corresponding to Weyl functions of $N \times N$ Jacobi matrices. We show that the restricted bracket has two systems of canonical coordinates on symplectic leafs of the foliation defined by level sets of the Casimir $\sum \lambda_k$. The existence of these two canonical coordinate systems is a consequence of invariance of the AH bracket under the group of linear-fractional transformations

$$w \to w' = \frac{aw+b}{cw+d}.$$

The first system of canonical coordinates is associated with N poles of $w(\lambda)$: $\lambda_0 < \cdots < \lambda_{N-1}$. The half of variables is $N-1$ points of the spectrum $\lambda_1, \ldots, \lambda_{N-1}$. Another half are the functions of $q(\lambda)$ at these points

$$\theta_k = \log\frac{(-1)^k q(\lambda_k)}{q(\lambda_0)}, \quad k = 1, \ldots, N-1.$$

This system is called action–angle coordinates. The verification of canonical relation is obtained by computing residues. The associated Hamiltonians are

$$H_j = \frac{1}{j}\sum \lambda_n^j, \quad j = 1, \ldots, N.$$

They produce the standard Toda flows which preserve the spectrum.

The second system is coming from $N-1$ finite roots $\gamma_1 < \cdots < \gamma_{N-1}$ of the equation

$$w(\gamma) = \frac{\rho_0}{\lambda_0 - \gamma} + \cdots + \frac{\rho_{N-1}}{\lambda_{N-1} - \gamma} = 0.$$

The adjoint $N-1$ variables are

$$\pi_k = \log(-1)^{N+k} p(\gamma_k), \quad k = 1, \ldots, N-1.$$

These are the divisor-quasimomentum coordinates. The Hamiltonians of this system are

$$T_j = \frac{1}{j}\sum \gamma_n^j, \quad j = 1, \dots, N-1.$$

These Hamiltonians produce the flows transversal to the isospectral manifolds. These flows preserve the divisor.

Now we will explain the origins of the divisor-quasimomentum coordinate system. Jacobi [13, Lecture 26], introduced his famous elliptic coordinates as finite roots $\gamma_0 < \gamma_1 < \cdots < \gamma_{N-1}$ of the equation

$$w(\gamma) = \frac{\rho_0}{\lambda_0 - \gamma} + \cdots + \frac{\rho_{N-1}}{\lambda_{N-1} - \gamma} = 1.$$

These N roots are considered to be the functions of N independent ρ's, while λ's remain fixed. In the case of finite Jacobi matrices this choice of parameters γ is bad. Due to the constraint $\sum \rho_k = 1$ they are functionally dependent. The same is true for any other constant instead of 1. Only for the special value 0 the equation $w(\gamma) = 0$ has one "unmovable" root $\gamma_0 = \infty$ and all other $N-1$ finite roots are functionally independent. Whence the divisor can be considered as a variant of the Jacobi elliptic coordinates.

The transformation $w(\lambda) \to w'(\lambda) = -1/w(\lambda)$ defines the dual Weyl function w'. The roots of the equations $w(\gamma) = 0$ become poles of the function $w'(\lambda)$. The invariance of the AH structure allows to establish canonical character of the divisor-quasimomentum coordinates again by computing residues.

We would like to conclude this section with the following remark. The traditional study of integrable dynamics is part of Hamiltonian mechanics with its standard objects like Poisson and symplectic manifolds, vector fields, differential forms, etc. We demonstrate in this paper that integrable dynamics on the space of Jacobi matrices or equivalently Weyl functions can be reformulated and studied purely in terms of complex analysis.

Organization of the paper. In Section 2, we review standard constructions of spectral theory of three-diagonal Jacobi matrices. In Section 3, we introduce the spectral shift function and establish the trace formulae. In Section 4, we consider the reducible Riemann surface and show how the Baker–Akhiezer function can be constructed using the orthogonal polynomial of the first kind and suitably normalized Weyl solution. In Section 5, we consider the representation of the Weyl function as a ratio of two polynomials and exponential representation. We demonstrate how using these representations one can construct the Abel map in the ordinary form and in the Baker form, correspondingly. We also describe the range of the map and show its' one-to-one character. This section completes the description of the spectral theory and its' relation to algebraic geometry. In Section 6, we introduce the Atiyah–Hitchin Poisson bracket and describe some of its' elementary properties. In Section 7, we construct canonical variables for the AH bracket. We also construct the Dirac restriction of the AH bracket on the submanifold of Weyl functions. We introduce two systems of canonical coordinates for the restricted bracket in Section 8. We also linearize the flow in terms of spectral shift function. Section 9 describes the isospectral and transversal flows.

2. The direct spectral problem

Most of the material of this section can be found in [1] and presented here in order to set notations. Consider a finite Jacobi matrix

$$L = \begin{bmatrix} v_0 & c_0 & 0 & \cdots & 0 \\ c_0 & v_1 & c_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & c_{N-3} & v_{N-2} & c_{N-2} \\ 0 & \cdots & 0 & c_{N-2} & v_{N-1} \end{bmatrix}.$$

It acts as a self-adjoint operator in the complex $l^2[0, N-1]$ with standard orthonormal basis $\delta(k) = \underbrace{(\dots, 1, \dots)}_{k\text{th place}}$, $k = 0, \dots, N-1$. The operator L has simple spectrum $\lambda_0 < \cdots < \lambda_{N-1}$ corresponding to normalized eigenvectors $e(\lambda_k) = (e_0(\lambda_k), \dots, e_{N-1}(\lambda_k))$, $k = 0, \dots, N-1$. Let $E(\lambda) = \sum_{\lambda_k<\lambda} e(\lambda_k) \otimes e(\lambda_k)$ be an orthogonal spectral measure of L. Thus,

$$L = \int \lambda \, \mathrm{d}E(z), \tag{2.1}$$

and

$$R(\lambda) = (L - \lambda I)^{-1} = \int \frac{\mathrm{d}E(z)}{z - \lambda}. \tag{2.2}$$

For any two vectors u, v the Parseval identity holds

$$(u, v) = \sum_k (u, e(\lambda_k))(v, e(\lambda_k)). \tag{2.3}$$

We associate with L the eigenvalue problem

$$v_0 y_0 + c_0 y_1 = \lambda y_0, \tag{2.4}$$

$$c_0 y_0 + v_1 y_1 + c_1 y_2 = \lambda y_1, \tag{2.5}$$

$$c_{n-1} y_{n-1} + v_n y_n + c_n y_{n+1} = \lambda y_n, \qquad n = 2, \dots, N-2,$$
$$c_{N-2} y_{N-2} + v_{N-1} y_{N-1} + c_{N-1} y_N = \lambda y_{N-1}. \tag{2.6}$$

The coefficient c_{N-1} is defined by the formula $c_{N-1} = \prod_{k=0}^{N-2} c_k^{-1}$. For the system (2.4)–(2.6) we introduce the solution

$$P(\lambda): \quad P_{-1}(\lambda) = 0, \qquad P_0(\lambda) = 1, \dots, P_N(\lambda),$$

and for the system (2.5) and (2.6)

$$Q(\lambda): \quad Q_0(\lambda) = 0, \qquad Q_1(\lambda) = \frac{1}{c_0}, \dots, Q_N(\lambda).$$

Let $L_{[k,p]}$ be the truncated matrix

$$L_{[k,p]} = \begin{bmatrix} v_k & c_k & \cdots & 0 \\ c_k & v_{k+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & c_{p-1} \\ \cdots & \cdots & c_{p-1} & v_p \end{bmatrix}. \tag{2.7}$$

Then for $n = 1, 2, \ldots, N-1$

$$P_n(\lambda) = (-1)^n \frac{\det(L_{[0,n-1]} - \lambda I)}{\prod_{k=0}^{n-1} c_k}, \tag{2.8}$$

and

$$Q_n(\lambda) = (-1)^{n+1} \frac{\det(L_{[1,n-1]} - \lambda I)}{\prod_{k=0}^{n-1} c_k}. \tag{2.9}$$

Now we will use the solutions P and Q to give formula (2.3) more concrete form.

For $P(\lambda) = (P_0(\lambda), \ldots, P_{N-1}(\lambda))$ from (2.8) we have

$$e(\lambda_k) = P(\lambda_k)\sqrt{\rho_k}, \qquad \rho_k = \frac{1}{\sum_{n=0}^{N-1} P_n^2(\lambda_k)}. \tag{2.10}$$

Thus (2.3) takes the form

$$(u, v) = \int \tilde{u}(\lambda)\tilde{v}(\lambda)\, \mathrm{d}\sigma(\lambda), \tag{2.11}$$

where

$$\tilde{u}(\lambda) = (u, P(\lambda)), \qquad \tilde{v}(\lambda) = (v, P(\lambda)), \qquad \mathrm{d}\sigma(\lambda) = \sum \delta(\lambda - \lambda_k)\rho_k.$$

Moreover, using (2.10) we have

$$(e(\lambda_k), \delta(0)) = \sqrt{\rho_k} > 0, \qquad \sum_{k=0}^{N-1} \rho_k = 1. \tag{2.12}$$

This implies, in particular, that $\delta(0)$ is a cyclic vector for L.

For the Weyl function defined as

$$w(\lambda) = -\frac{Q_N(\lambda)}{P_N(\lambda)}$$

formulae (2.8) and (2.9) imply

$$w(\lambda) = -\frac{(-1)^{N+1}\det(L_{[1,N-1]} - \lambda I)}{(-1)^N \det(L - \lambda I)} = -\frac{(-1)^{N+1}\prod_{s=1}^{N-1}(\gamma_s - \lambda)}{(-1)^N \prod_{n=0}^{N-1}(\lambda_n - \lambda)}, \tag{2.13}$$

where the roots λ and γ interlace

$$\lambda_0 < \gamma_1 < \lambda_1 < \cdots < \lambda_{N-2} < \gamma_{N-1} < \lambda_{N-1} \tag{2.14}$$

due to the Sturm theorem.

By construction $Q_N + wP_N = 0$ for all λ. Formulae (2.4)–(2.6) produce $(L - \lambda I)(Q + wP) = \delta(0)$ and $Q + wP = R(\lambda)\delta(0)$. Formula (2.2) implies

$$w(\lambda) = (R(\lambda)\delta(0), \delta(0)) = \int \frac{(\delta(0), \mathrm{d}E(z)\delta(0))}{z - \lambda} = \int \frac{\mathrm{d}\sigma(z)}{z - \lambda}. \tag{2.15}$$

From (2.1) for the moments of the measure $\mathrm{d}\sigma$ we have

$$s_k = (L^k\delta(0), \delta(0)) = \int \lambda^k \,\mathrm{d}\sigma(\lambda). \tag{2.16}$$

Using (2.12),

$$w(\lambda) = -\sum_{n=0}^{\infty} s_n \lambda^{-(n+1)}, \quad \text{where } s_0 = 1. \tag{2.17}$$

We conclude this section with derivation of trace formulae. To simplify notations we assume that $\lambda_0 = 0$. Then, (2.13) becomes

$$w(\lambda) = -\frac{1}{\lambda}\prod_{k=1}^{N-1}\left(\frac{\gamma_k - \lambda}{\lambda_k - \lambda}\right). \tag{2.18}$$

After simple algebra,

$$\frac{\gamma_k - \lambda}{\lambda_k - \lambda} = 1 + \sum_{p=1}^{\infty} \frac{\lambda_k^{p-1}(\lambda_k - \gamma_k)}{\lambda^p} = \sum_{p=0}^{\infty} \Delta_k^p \lambda^{-p}, \quad \text{where } \Delta_k^0 = 1.$$

Furthermore,

$$w(\lambda) = -\sum_{n=0}^{\infty}\left[\sum_{p_1+\cdots+p_{N-1}=n}\ \prod_{k=1}^{N-1} \Delta_k^{p_k}\right]\lambda^{-(n+1)}.$$

Comparing it with (2.17) we obtain the trace formulae

$$s_n = \sum_{p_1+\cdots+p_{N-1}=n}\ \prod_{k=1}^{N-1} \Delta_k^{p_k}. \tag{2.19}$$

The first few are listed below

$$s_1 = \sum_k \Delta_k^1, \qquad s_2 = \sum_k \Delta_k^2 + \sum_{k_1 \neq k_2} \Delta_{k_1}^1 \Delta_{k_2}^1,$$
$$s_3 = \sum_k \Delta_k^3 + 2\sum_{k_1 \neq k_2} \Delta_{k_1}^2 \Delta_{k_2}^1 + \sum_{k_1 \neq k_2 \neq k_3} \Delta_{k_1}^1 \Delta_{k_2}^1 \Delta_{k_3}^1, \ldots .$$

To derive the standard trace formulae [14] from the resolvent expansion

$$R(\lambda) = (L - \lambda I)^{-1} = -\frac{I}{\lambda} - \frac{L}{\lambda^2} - \frac{L^2}{\lambda^3} - \cdots,$$

we obtain

$$w(\lambda) = -\frac{(I\delta(0), \delta(0))}{\lambda} - \frac{(L\delta(0), \delta(0))}{\lambda^2} - \frac{(L^2\delta(0), \delta(0))}{\lambda^3} - \cdots,$$
$$w(\lambda) = -\frac{1}{\lambda} - \frac{v_0}{\lambda^2} - \frac{v_0^2 + c_0^2}{\lambda^3} - \cdots. \tag{2.20}$$

Matching the coefficients in (2.17) and (2.20),

$$v_0 = \sum_k \Delta_k^1, \qquad c_0^2 = \sum_k \Delta_k^2 - \sum_k (\Delta_k^1)^2, \ldots .$$

3. The trace formulae via Krein spectral shift

We assume $\lambda_0 = 0$, then (2.13) becomes

$$w(\lambda) = -\frac{1}{\lambda}\frac{\det(L_{[1,N-1]} - \lambda I)}{\det(L|\mathrm{Ker}\, L^{\perp} - \lambda I)}, \tag{3.1}$$

where $L|\mathrm{Ker}\, L^{\perp}$ is a restriction of L on the orthogonal compliment to Ker L. By elementary transformations the ratio of two determinants can be put in the form

$$w(\lambda) = -\frac{1}{\lambda}\prod_{s=1}^{N-1}\left(\frac{\gamma_s - \lambda}{\lambda_s - \lambda}\right) = -\frac{1}{\lambda}\exp\sum_{s=1}^{N-1}\int_{\lambda_s}^{\gamma_s}\frac{\mathrm{d}z}{z-\lambda}$$
$$= -\frac{1}{\lambda}\exp(\Xi(\lambda)) = -\frac{1}{\lambda}\exp\int\frac{\xi(z)}{z-\lambda}\,\mathrm{d}z, \tag{3.2}$$

where[1]

$$\xi(z) = n_{L|\mathrm{Ker}\, L^{\perp}}(z) - n_{L_{[1,N-1]}}(z)$$

is the Krein spectral shift function [17]. This exponential representation of the Weyl function has much wider range of applicability then formula (3.1), which requires separate existence of determinants in the numerator and denominator. It can be obtained, for example, for infinite unbounded matrices under very mild condition on closeness of $L_{[1,\infty]}$ and $L|\mathrm{Ker}\, L^{\perp}$.

One can obtain trace formulae in terms of $f_k = \int z^k \xi(z)\,\mathrm{d}z$ entering into the asymptotic expansion

$$\Xi(\lambda) = -\sum_{n=0}^{\infty} f_n \lambda^{-(n+1)}.$$

Expanding the exponent in (3.2) and matching the coefficients with (2.17)

$$s_1 = -f_0, \qquad s_2 = \frac{f_0^2}{2} - f_1, \qquad s_3 = f_0 f_1 - f_2 - \frac{f_0^3}{6}, \ldots .$$

[1] $n_L(z)$ is a counting function, $n_L(z) = \#\{\text{eigenvalues of } L \le z\}$.

Evidently, these formulae can be put in the form (2.19) using representation of $\xi(z)$ as a difference of two counting functions.

4. The spectral curve: the Baker–Akhiezer function

The function $w(\lambda)$ determines the matrix L or in another words the functions $w(\lambda) = w(\lambda, L)$ are coordinates on the space $\mathcal{L}$ of all $N \times N$ Jacobi matrices. The "index" λ which labels the "coordinates" takes the values in $C^1 \setminus \mathrm{spec}\, L$. This statement goes back to Stieltjes [25]. There are two standard ways to recover L from w. The first is to expand $w(\lambda)$ into continued fraction, from which one can read the coefficient of L. The second is to construct polynomials orthogonal with respect to the spectral measure recovered from $w(\lambda)$. A three term recurrent relation for these polynomials is, in fact, the matrix L. Recently, the classical inversion problem, received a new, algebro-geometrical solution [18]. The main novel part of Krichever and Vaninsky [18] is, the so-called, Baker–Akhiezer function for a *reducible* curve. This construction is described below.

We start with the standard Weyl solution $Q + wP$ and note that $(Q + wP)/w$ is a solution of (2.5) and (2.6) which is equal to 1 at $n = 0$ and vanishes at $n = N$ for all λ. The vector P is also a solution of (2.5) and (2.6) which is equal to 1 for $n = 0$ and all values of λ and vanishes at $n = N$ for $\lambda = \lambda_k$. Thus we have a "gluing" condition

$$P(\lambda_k) = \frac{Q + wP}{w}(\lambda_k). \tag{4.1}$$

In other words, at the points of the spectrum the function $w(\lambda)$ conjugates two solutions P and $Q + wP$ which vanish at the left ($n = -1$) or right ($n = N$) correspondingly.

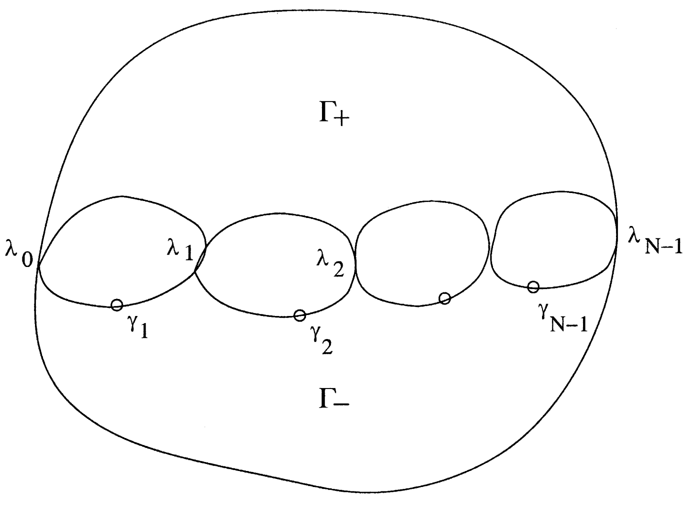

Fig. 1. Riemann surface.

The singular algebraic curve Γ (Fig. 1) is obtained by gluing at the points of the spectrum two copies of the complex plane. Define Baker–Akhiezer function on Γ by the formula, $\lambda = \lambda(\epsilon)$:

$$\psi(\epsilon) = \begin{cases} P(\lambda) & \text{if } \epsilon \in \Gamma_+, \\ \dfrac{Q + wP}{w(\lambda)} & \text{if } \epsilon \in \Gamma_-. \end{cases}$$

The function ψ is continuous on Γ due to the gluing condition (4.1). The BA function has the only simple poles at the points of the divisor $(\gamma_1, -), \ldots, (\gamma_{N-1}, -)$. At two infinities the BA function has poles of prescribed order. These data determine the BA function and therefore the operator L. Thus, the uniqueness statement can be viewed as a consequence of the Riemann–Roch theorem. This remark completes the description of the direct spectral problem. The inverse problem can be solved using an explicit formula for the time-dependent BA function (for details, see [18]).

5. The Abel map and the Jacobian

Let Rat_N be a set of all rational functions which map the upper half-plane into itself, vanish at infinity and have N poles. Any function $w(\lambda)$ from Rat_N has the form

$$w(\lambda) = \sum_{k=0}^{N-1} \frac{\rho_k}{\lambda_k - \lambda} \tag{5.1}$$

with real poles at $\gamma_0 < \cdots < \lambda_{N-1}$ and $\rho_k > 0$. For λ real below/above the spectrum the function $w(\lambda)$ is positive/negative. Furthermore,

$$w'(\lambda) = \sum \frac{\rho_k}{(\lambda_k - \lambda)^2} > 0,$$

and $w(\lambda)$ continuously changes from minus infinity to plus infinity, when λ runs between two consecutive poles. Thus the function $w(\lambda)$ has exactly $N-1$ zeros γ's which interlace λ's as in formula (2.14).

Furthermore, any function from Rat_N can be represented as a ratio of two polynomials

$$\begin{aligned} w(\lambda) &= -\frac{q(\lambda)}{p(\lambda)} = -\frac{q_0(-1)^{N+1}\prod_{s=1}^{N-1}(\gamma_s - \lambda)}{(-1)^N \prod_{n=0}^{N-1}(\lambda_n - \lambda)} \\ &= -\frac{q_0\lambda^{N-1} + q_1\lambda^{N-2} + \cdots + q_{N-1}}{\lambda^N + p_0\lambda^{N-1} + \cdots + p_{N-1}}. \end{aligned} \tag{5.2}$$

The polynomials are defined up to a multiple factor. In formula (5.2) it is chosen such that the leading coefficient of the denominator is 1, similar to (2.13). Evidently, the polynomial $q(\lambda)$ can be determined from its' values $q(\lambda_0), \ldots, q(\lambda_{N-1})$ which are *free* parameters.

Now we turn to the submanifold Rat'_N with $\sum \rho_k = 1$. These are the functions from Rat_N which are Weyl functions of finite Jacobi matrices. The values $q(\lambda_0), \ldots, q(\lambda_{N-1})$

are not independent anymore. Indeed, from the identity

$$-\frac{q(\lambda)}{p(\lambda)} = \sum \frac{\rho_n}{\lambda_n - \lambda},$$

and condition $p(\lambda_n) = 0$ we obtain $q(\lambda_k) = p'(\lambda_k)\rho_k$. Therefore,

$$\sum_{n=0}^{N-1} \frac{q(\lambda_n)}{p'(\lambda_n)} = 1.$$

Due to the relation $q_0 = \sum \rho_k$, an another way to say that $w(\lambda) \in \mathrm{Rat}'_N$ is that the polynomial $q(\lambda)$ in (5.2) is a monic polynomial.

For a function $w(\lambda) \in \mathrm{Rat}'_N$ we define angle variables by the formula

$$\theta_k = \log \frac{(-1)^k q(\lambda_k)}{q(\lambda_0)}, \quad k = 1, \dots, N-1. \tag{5.3}$$

There are exactly k roots of $q(\lambda)$ between λ_0 and λ_k and it changes sign k times when λ varies from λ_0 to λ_k. Whence variables θ's are always real.

To clarify geometrical meaning of the variables θ's we introduce (Fig. 2) a standard homology basis on the curve Γ corresponding to some Jacobi matrix. We define differentials ω_k by the formula

$$\omega_k = \left[\frac{1}{z-\lambda_k} - \frac{1}{z-\lambda_0}\right] \mathrm{d}z, \quad k = 1, \dots, N-1.$$

Evidently, the normalization condition holds

$$\int_{a_p} \omega_k = 2\pi \mathrm{i}\delta_k^p, \quad k, p = 1, \dots, N-1,$$

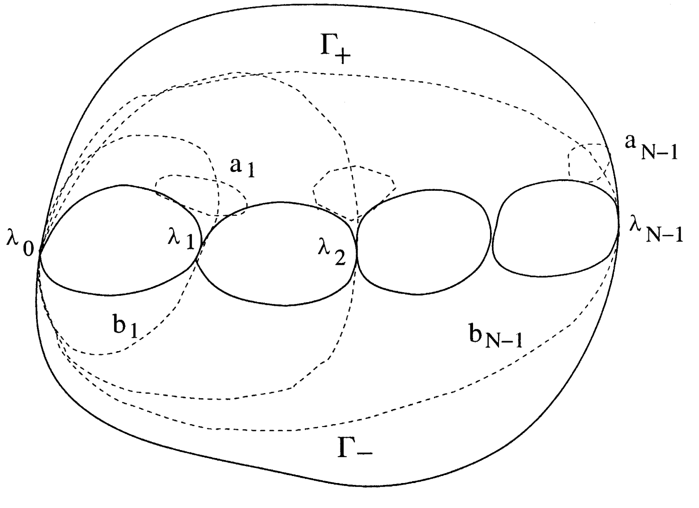

Fig. 2. Basis of circles.

while b-periods of ω's are real and infinite. For the variables θ, we have

$$\begin{aligned}\theta_k &= \pi \mathrm{i}k + \log q(\lambda_k) - \log q(\lambda_0)\\ &= \pi \mathrm{i}k + \sum_{s=1}^{N-1} \int_{\infty-}^{\gamma_s} \left[\frac{1}{z-\lambda_k} - \frac{1}{z-\lambda_0}\right] \mathrm{d}z = \pi \mathrm{i}k + \sum_{s=1}^{N-1} \int_{\infty-}^{\gamma_s} \omega_k. \qquad (5.4)\end{aligned}$$

Therefore, θ's are the values of the Abel map from the divisor $\gamma_1, \ldots, \gamma_{N-1}$ into R^{N-1}, the noncompact real part of the Jacobian. It will be shown that this map is onto.

Fix some matrix L_0. All matrices with the same spectrum $\lambda_0 < \cdots < \lambda_{N-1}$ as L_0 constitute a spectral class of L_0, which we denote by $\mathcal{S}(L_0)$. The spectral class $\mathcal{S}(L_0)$ is in 1:1 correspondence with Weyl functions from Rat'_N

$$w(\lambda) = \sum_{n=0}^{N-1} \frac{\rho_n}{\lambda_n - \lambda}, \quad \sum \rho_n = 1.$$

Theorem 1.

(i) *The variables $\gamma_1, \ldots, \gamma_{N-1}$ are coordinates on $\mathcal{S}(L_0)$. Any sequence of γ's which occupies open segments $\lambda_{k-1} < \gamma_k < \lambda_k$, $k = 1, \ldots, N-1$ corresponds to some matrix from $\mathcal{S}(L_0)$.*

(ii) *The variables $\theta_1, \ldots, \theta_{N-1}$ are coordinates on $\mathcal{S}(L_0)$. Any sequence of θ's from R^{N-1} corresponds to some matrix from $\mathcal{S}(L_0)$.*

Proof.

(i) The variables γ's determine the roots of $q(\lambda)$ and therefore the function $w(\lambda)$. Whence, γ's are coordinates.

To prove that γ's are free pick any sequence of γ's and form $q(\lambda)$. Then, by Lagrange interpolation

$$-\frac{q(\lambda)}{p(\lambda)} = \sum_{n=0}^{N-1} \frac{q(\lambda_n)}{p'(\lambda_n)} \frac{1}{\lambda_n - \lambda}.$$

It is easy to check all $\rho_n = q(\lambda_n)/p'(\lambda_n)$ are strictly positive. It remains to prove that $\sum \rho_n = 1$. Indeed, the formula

$$q(\lambda) = \sum_{n=0}^{N-1} \frac{q(\lambda_n)}{p'(\lambda_n)} \frac{p(\lambda)}{\lambda - \lambda_n}$$

implies

$$1 = \lim_{\lambda\to\infty} \frac{q(\lambda)}{\lambda^{N-1}} = \lim_{\lambda\to\infty} \sum \frac{q(\lambda_n)}{p'(\lambda_n)} \frac{p(\lambda)}{\lambda^{N-1}(\lambda - \lambda_n)} = \sum \frac{q(\lambda_n)}{p'(\lambda_n)}.$$

We are done.

(ii) From the definition of θ's for any $k = 1, \dots, N-1$:

$$e^{\theta_k} = (-1)^k \frac{q(\lambda_k)}{q(\lambda_0)}.$$

Using $q(\lambda_k) = p'(\lambda_k)\rho_k$ we have

$$\rho_k = \rho_0 \, e^{\theta_k} (-1)^k \frac{p'(\lambda_0)}{q(\lambda_0)}.$$

It is easy to check that all $\rho_k > 0$. There exists just one ρ_0 such that $\sum \rho_n = 1$. This implies that θ's are coordinates and they are free. The theorem is proved. □

The fact that the isospectral set $\mathcal{S}(L_0)$ is a diffeomorphic to R^{N-1} was already noted by Moser [21]. Tomei [26] then showed that $\mathcal{S}(L_0)$ can be compactified and it becomes a convex polyhedron. The symplectic interpretation of this result as a version of the Atiyah–Guillemin–Sternberg convexity theorem was given by Bloch et al. [4].

Now for a function $w(\lambda) \in \mathrm{Rat}'_N$ with $\lambda_0 = 0$ we consider the exponential representation (3.2)

$$w(\lambda) = -\frac{1}{\lambda} \exp(\varXi(\lambda)),$$

and define another set of angles by the formula

$$\theta'_k = \lim_{\lambda \to \lambda_k} \left[\varXi(\lambda) - \varXi(0) + \log \frac{\lambda_k - \lambda}{\lambda_k} \right] + \pi \mathrm{i}, \quad k = 1, \dots, N-1. \tag{5.5}$$

This formula can be put in the form

$$\theta'_k = \sum_{s=1, s\neq k}^{N-1} \int_{\lambda_s}^{\gamma_s} \omega_k + \int_{\infty-}^{\gamma_k} \omega_k + \pi \mathrm{i}.$$

Whence, θ'_k correspond to the Abel sum in the Baker form. The regularization is necessary, because the term $\int_{\lambda_k}^{\gamma_k} \omega_k$ diverges logarithmically on singular curve Γ. Furthermore, we have simple relation

$$\theta_k = \theta'_k + \pi \mathrm{i}(k-1) + \log \prod_{s=1, s\neq k}^{N-1} \frac{\lambda_s - \lambda_k}{\lambda_s}. \tag{5.6}$$

The angles θ and θ' differ by the real quantity which depend on the curve *only*. Evidently, the variables θ' are coordinates on $S(L_0)$ and their range is R^{N-1}.

6. The Poisson bracket on Weyl functions

Following [15], we consider functions $w(\lambda)$ with the properties: (i) analytic in the half-planes $\Im z > 0$ and $\Im z < 0$. (ii) $w(\bar{z}) = \overline{w(z)}$ if $\Im z \neq 0$. (iii) $\Im w(z) > 0$ if $\Im z > 0$. All such functions are called R-functions. They play a central role in the theory of the resolvent of self-adjoint operators. The Weyl function of a Jacobi matrix is an R-function.

Atiyah and Hitchin [2] introduced a Poisson structure on the space of rational functions. In the recent paper [9], Faybusovich and Gehtman wrote Atiyah–Hitchin and higher Poisson structures on rational functions in compact invariant form. They defined the Atiyah–Hitchin bracket by the formula

$$\{w(\lambda), w(\mu)\} = \frac{(w(\lambda) - w(\mu))^2}{\lambda - \mu}. \tag{6.1}$$

Here we discuss some of its' remarkable properties [27].

We think about $w(\lambda)$ as an element of some commutative complex algebra which depends holomorphically on the parameter λ. Evidently, (6.1) is skew-symmetric with respect to λ and μ. It is natural and require linearity of the bracket

$$\{aw(\lambda) + bw(\lambda), w(\nu)\} = a\{w(\lambda), w(\nu)\} + b\{w(\lambda), w(\nu)\}, \tag{6.2}$$

where a and b are constants. The symbol $w(\lambda)$ for λ inside the contour C is given by the Cauchy formula

$$w(\lambda) = \frac{1}{2\pi \mathrm{i}} \int_C \frac{w(\zeta)}{\zeta - \lambda}\, \mathrm{d}\zeta.$$

Whence due to (6.2) the values of the bracket in different points are related

$$\{w(\lambda), w(\mu)\} = \frac{1}{2\pi \mathrm{i}} \int_C \frac{\{w(\zeta), w(\mu)\}}{\zeta - \lambda}\, \mathrm{d}\zeta.$$

It can be verified that the bracket (6.1) satisfies this compatibility condition.

Also, it is natural to require for the bracket the Leibnitz rule

$$\{w(\lambda)w(\mu), w(\nu)\} = w(\lambda)\{w(\mu), w(\nu)\} + w(\mu)\{w(\lambda), w(\nu)\}. \tag{6.3}$$

It can be verified in a long but simple calculation that (6.2) and (6.3) imply the Jacobi identity

$$\{w(\lambda), \{w(\mu), w(\nu)\}\} + \{w(\mu), \{w(\nu), w(\lambda)\}\} + \{w(\nu), \{w(\lambda), w(\mu)\}\} = 0.$$

The particularly useful to us is an invariance of (6.1) under the group of linear-fractional transformations

$$w \to w' = \frac{aw + b}{cw + d}, \tag{6.4}$$

where $a, b, c,$ and d are constants. This property will be used in the construction of the second system of canonical coordinates.

In our study of finite Jacobi matrices we need a small subclass $\mathrm{Rat}_N \subset R$. These are the functions given by formula (5.1). All such functions have asymptotic expansion at infinity

$$w(\lambda) = -\frac{s_0}{\lambda} - \frac{s_1}{\lambda^2} - \cdots .$$

We consider submanifold Rat'_N with $s_0 = 1$, or equivalently, $\sum \rho_k = 1$. We will demonstrate that the Dirac restriction of the bracket (6.1) to this submanifold takes the form

$$\{w(\lambda), w(\mu)\}' = (w(\lambda) - w(\mu))\left(\frac{w(\lambda) - w(\mu)}{\lambda - \mu} - w(\lambda)w(\mu)\right). \tag{6.5}$$

The linear Poisson structure on the phase space $\mathcal{L}$ is defined by the formulae

$$\{c_k, v_k\} = -\frac{c_k}{2}, \qquad \{c_k, v_{k+1}\} = \frac{c_k}{2}, \tag{6.6}$$

and all other brackets vanish. Whence, the linear bracket (6.6) corresponds to the restriction (6.5) of the AH structure on the submanifold Rat'_N.

Formula (6.5) can be used to define the Poisson structure (6.6). For example, substituting (2.20) into (6.5), after simple algebra we obtain

$$\frac{2c_0\{v_0, c_0\}}{\lambda^2\mu^2}\left(\frac{1}{\mu} - \frac{1}{\lambda}\right) + \cdots = \frac{c_0^2}{\lambda^2\mu^2}\left(\frac{1}{\mu} - \frac{1}{\lambda}\right) + \cdots.$$

From this one can read the first identity: $\{c_0, v_0\} = -c_0/2$.

A construction of canonical coordinates for the bracket (6.1) or (6.5) will be given in terms of various representations for Rat_N and Rat'_N.

7. Canonical coordinates on Rat_N: the Dirac reduction

We start with the construction of the first system of canonical coordinate on Rat_N for the bracket (6.1). The next theorem shows that the parameters

$$\lambda_0, \ldots, \lambda_{N-1}, \qquad \rho_0, \ldots, \rho_{N-1}$$

in formula (5.1) are "almost" canonically paired.

Theorem 2. *The bracket (6.1) in λ–ρ coordinates has the form*

$$\{\rho_k, \rho_n\} = \frac{2\rho_k\rho_n}{\lambda_n - \lambda_k}(1 - \delta_k^n), \tag{7.1}$$

$$\{\rho_k, \lambda_n\} = \rho_k\delta_k^n, \tag{7.2}$$

$$\{\lambda_k, \lambda_n\} = 0. \tag{7.3}$$

Proof. We represent ρ's and λ's as contour integrals

$$\rho_k = -\frac{1}{2\pi \mathrm{i}}\int_{O_k} w(\zeta)\,\mathrm{d}\zeta, \qquad \rho_k\lambda_k = -\frac{1}{2\pi \mathrm{i}}\int_{O_k} \zeta w(\zeta)\,\mathrm{d}\zeta.$$

In both formulae the contour O_k surrounding λ_k is traversed counterclockwise. Therefore, for $k \neq n$

$$\begin{aligned}
\{\rho_k, \rho_n\} &= \left\{\frac{1}{2\pi \mathrm{i}} \int_{O_k} w(\zeta)\,\mathrm{d}\zeta, \frac{1}{2\pi \mathrm{i}} \int_{O_n} w(\eta)\,\mathrm{d}\eta\right\} = \frac{1}{(2\pi \mathrm{i})^2} \int_{O_k} \int_{O_n} \{w(\zeta), w(\eta)\}\,\mathrm{d}\zeta\,\mathrm{d}\eta \\
&= \frac{1}{(2\pi \mathrm{i})^2} \int_{O_k} \int_{O_n} \frac{(w(\zeta)-w(\eta))^2}{\zeta-\eta}\,\mathrm{d}\zeta\,\mathrm{d}\eta \\
&= \frac{1}{2\pi \mathrm{i}} \int_{O_k} \mathrm{d}\zeta\, w^2(\zeta) \left[\frac{1}{2\pi \mathrm{i}} \int_{O_n} \mathrm{d}\eta\, \frac{1}{\zeta-\eta}\right] - \frac{2}{(2\pi \mathrm{i})^2} \int_{O_k} \int_{O_n} \frac{w(\zeta)w(\eta)}{\zeta-\eta}\,\mathrm{d}\zeta\,\mathrm{d}\eta \\
&\quad + \frac{1}{2\pi \mathrm{i}} \int_{O_n} \mathrm{d}\eta\, w^2(\eta) \left[\frac{1}{2\pi \mathrm{i}} \int_{O_k} \mathrm{d}\zeta\, \frac{1}{\zeta-\eta}\right]. \qquad (7.4)
\end{aligned}$$

The two terms with square brackets vanish and for (7.4) we obtain

$$-\frac{2}{2\pi \mathrm{i}} \int \mathrm{d}\zeta\, w(\zeta) \left[\frac{1}{2\pi \mathrm{i}} \int \mathrm{d}\eta\, \frac{w(\eta)}{\zeta-\eta}\right]. \qquad (7.5)$$

For ζ in the exterior of the contour O_p, we have

$$\frac{1}{2\pi \mathrm{i}} \int_{O_p} \mathrm{d}\eta\, \frac{w(\eta)}{\zeta-\eta} = \frac{\rho_p}{\lambda_p-\zeta}.$$

Applying this formula twice to (7.5) we obtain (7.1). If $k = n$ then similar arguments show that the bracket (7.1) vanishes.

To prove (7.2) we compute for $k \neq n$

$$\{\lambda_k\rho_k, \rho_n\} = \frac{2\lambda_k\rho_k\rho_n}{\lambda_n-\lambda_k}.$$

From another side,

$$\{\lambda_k\rho_k, \rho_n\} = \lambda_k\{\rho_k, \rho_n\} + \rho_k\{\lambda_k, \rho_n\}.$$

This together with (7.1) imply that the bracket $\{\lambda_k, \rho_n\}$ vanish. For $k = n$ we have $\{\lambda_k\rho_k, \rho_n\} = -\rho_k^2$ and $\{\rho_k, \lambda_k\} = \rho_k$. The formula (7.2) is proved.

To prove (7.3) for $k \neq n$ we compute

$$\{\lambda_k\rho_k, \lambda_n\rho_n\} = \frac{2\lambda_k\rho_k\lambda_n\rho_n}{\lambda_n-\lambda_k}.$$

From another side,

$$\{\lambda_k\rho_k, \lambda_n\rho_n\} = \rho_k\rho_n\{\lambda_k, \lambda_n\} + \rho_k\lambda_n\{\lambda_k, \rho_n\} + \rho_n\lambda_k\{\rho_k, \lambda_n\} + \lambda_k\lambda_n\{\rho_k, \rho_n\}.$$

This together with formulae (7.1)–(7.3) imply that the bracket $\{\lambda_k, \lambda_n\}$ vanishes. For $k = n$ arguments are the same. The proof is finished. □

Therefore, for the bracket (6.1) on Rat_N canonical coordinates are associated with poles of $w(\lambda)$

$$\lambda_0, \ldots, \lambda_{N-1}, \qquad q(\lambda_0), \ldots, q(\lambda_{N-1}).$$

Indeed, from the identity

$$-\frac{q(\lambda)}{p(\lambda)} = \sum \frac{\rho_n}{\lambda_n - \lambda},$$

and condition $p(\lambda_n) = 0$ we obtain $q(\lambda_k) = p'(\lambda_k)\rho_k$. Furthermore, using (7.2) and (7.3)

$$\{q(\lambda_k), \lambda_n\} = \{p'(\lambda_k)\rho_k, \lambda_n\} = p'(\lambda_k)\rho_k\delta_k^n = q(\lambda_k)\delta_k^n.$$

All other brackets vanish

$$\{q(\lambda_k), q(\lambda_n)\} = \{\lambda_k, \lambda_n\} = 0.$$

In this coordinate form the Poisson structure on Rat_N was introduced in [2]. These identities imply

$$\{q(\lambda), q(\mu)\} = \{p(\lambda), p(\mu)\} = 0,$$

and

$$\{q(\lambda), p(\mu)\} = \frac{q(\lambda)p(\mu) - q(\mu)p(\lambda)}{\lambda - \mu}.$$

The last expression is called a Bezontian, see [16]. This form of the bracket easily leads to (6.1).

The second set of canonical coordinates on Rat_N is associated with zeros of $w(\lambda)$

$$\gamma_1, \ldots, \gamma_{N-1}, \qquad p(\gamma_1), \ldots, p(\gamma_{N-1}), \qquad q_0, p_0.$$

To prove this, we introduce the new "dual" function $w'(\lambda)$ as an inverse of function (5.2)

$$w'(\lambda) = -\frac{1}{w(\lambda)} = \frac{p(\lambda)}{q(\lambda)}.$$

Due to (6.4) the bracket for the dual function is given by formula (6.1). The new meromorphic function maps the upper half-plane into itself and has the expansion

$$w'(\lambda) = \frac{\lambda}{q_0} + c + \sum_{s=1}^{N-1} \frac{\rho'_s}{\gamma_s - \lambda},$$

where

$$c = \frac{p_0 q_0 - q_1}{q_0^2} = \frac{p_0}{q_0} + \frac{\sum \gamma_s}{q_0}, \qquad \rho'_s > 0.$$

Theorem 3. *The following identities hold*:

$$\{\rho'_k, \rho'_n\} = \frac{2\rho'_k\rho'_n}{\gamma_n - \gamma_k}(1 - \delta_k^n), \tag{7.6}$$

$$\{\rho'_k, \gamma_n\} = \rho'_k\delta_k^n, \tag{7.7}$$

$$\{\gamma_k, \gamma_s\} = 0, \tag{7.8}$$

$$\{q_0, \rho'_k\} = \{q_0, \gamma_s\} = 0, \tag{7.9}$$

$$\{\rho'_k, p_0\} = \rho'_k, \tag{7.10}$$

$$\{p_0, \gamma_s\} = 0, \tag{7.11}$$

$$\{p_0, q_0\} = q_0. \tag{7.12}$$

Proof. The identities (7.6)–(7.8) using integral representation

$$\rho'_k = -\frac{1}{2\pi \mathrm{i}} \int_{O_k} w'(\zeta)\,\mathrm{d}\zeta, \qquad \rho'_k \gamma_k = -\frac{1}{2\pi \mathrm{i}} \int_{O_k} \zeta w'(\zeta)\,\mathrm{d}\zeta$$

can be proved exactly the same way as identities (7.1)–(7.3) of Theorem 2.

Let us compute the first bracket (7.9)

$$\begin{aligned}\{q_0, \rho'_k\} &= \left\{\lim_{\lambda\to\infty} \frac{\lambda}{w'(\lambda)}, -\frac{1}{2\pi \mathrm{i}} \int_{O_k} w'(\zeta)\,\mathrm{d}\zeta\right\} \\ &= \lim_{\lambda\to\infty} \frac{\lambda}{2\pi \mathrm{i} \omega'(\lambda)^2} \int_{O_k} \frac{(w'(\lambda) - w'(\zeta))^2}{\lambda - \zeta}\,\mathrm{d}\zeta.\end{aligned}$$

Expanding the square and computing each term separately we see that the bracket vanishes. The proof of the second identity (7.9) is exactly the same.

To prove (7.10) we note

$$\{c, \rho'_k\} = \frac{\{p_0, \rho'_k\}}{q_0} - \frac{\rho'_k}{q_0}.$$

From another hand

$$\begin{aligned}\{c, \rho'_k\} &= \left\{\lim_{\lambda\to\infty} w'(\lambda) - \frac{\lambda}{q_0}, -\frac{1}{2\pi \mathrm{i}} \int_{O_k} w'(\zeta)\,\mathrm{d}\zeta\right\} \\ &= \lim_{\lambda\to\infty} -\frac{1}{2\pi \mathrm{i}} \int_{O_k} \frac{(w'(\lambda) - w'(\zeta))^2}{\lambda - \zeta}\,\mathrm{d}\zeta = -\frac{2\rho'_k}{q_0}.\end{aligned}$$

Comparing it with the previous formula we obtain (7.10). The proof of formula (7.11) is similar.

To prove the last formula (7.12) we compute

$$\begin{aligned}\frac{\{p_0, q_0\}}{q_0} &= \{c, q_0\} = \left\{\lim_{\lambda\to\infty} w'(\lambda) - \frac{\lambda}{q_0}, q_0\right\} = \lim_{\lambda\to\infty} \left\{w'(\lambda), \lim_{\mu\to\infty} \frac{\mu}{w'(\mu)}\right\} \\ &= \lim_{\lambda\to\infty} \lim_{\mu\to\infty} -\frac{\mu}{w'(\mu)^2} \frac{(w'(\lambda) - w'(\mu))^2}{\lambda - \mu} = 1.\end{aligned}$$

This implies the result. The proof is finished.

From (7.7) using the formula $p(\gamma_s) = -\rho_s' q'(\gamma_s)$ we obtain

$$\{p(\gamma_n), \gamma_k\} = p(\gamma_n)\delta_n^k.$$

From (7.6), (7.7) and (7.9)

$$\{p(\gamma_n), q(\gamma_k)\} = 0.$$

From (7.9)

$$\{p(\gamma_n), q_0\} = 0.$$

From (7.10) and (7.12)

$$\{p(\gamma_n), p_0\} = 0.$$

These identities together with identities of Theorem 3 provide a proof of our statement.

This coordinate system is useful in construction of the Dirac restriction [8], of the AH bracket (6.1) on the submanifold $M \subset \mathrm{Rat}_N$ determined by the conditions

$$\Phi_1 = p_0 = c_1, \qquad \Phi_2 = \log q_0 = c_2,$$

where c_1 and c_2 are some real constants.

Consider a more general problem. The submanifold M of dimension $2N-m$ is determined by the conditions

$$\Phi_1 = c_1, \Phi_2 = c_2, \ldots, \Phi_m = c_m,$$

where Φ's are some functions on the phase space and c's are real constants. The bracket $\{\bullet, \bullet\}$ is modified

$$\{F_1, F_2\}' = \{F_1, F_2\} + \sum_{k=1}^{m} \sigma_k \{F_1, \Phi_k\}$$

with σ's chosen such that

$$\Phi_k^{\bullet} = \{\Phi_k, F_2\} + \sum_{s=1}^{m} \sigma_s \{\Phi_k, \Phi_s\} = 0$$

for all $k = 1, \ldots, m$. Geometrically, this condition means that the vector fields produced in the bracket $\{\bullet, \bullet\}'$ are tangent to M. If the matrix $\|\{\Phi_k, \Phi_s\}\|$ has an inverse $\|C_{ks}\|$, then the last system can be solved for σ's and

$$\{F_1, F_2\}' = \{F_1, F_2\} + \sum_{k,s=1}^{m} \{F_1, \Phi_k\} C_{ks} \{F_2, \Phi_s\}.$$

Implementing this procedure for our choice of functionals Φ_1 and Φ_2 we obtain

$$\sigma_1 = \{\log q_0, F_2\}, \qquad \sigma_2 = -\{p_0, F_2\},$$

and

$$\{F_1, F_2\}' = \{F_1, F_2\} + \{\log q_0, F_2\}\{F_1, p_0\} - \{p_0, F_2\}\{F_1, \log q_0\}.$$

One can easily verify that

$$\{q(\lambda), q(\mu)\}' = \{p(\lambda), p(\mu)\}' = 0.$$

Using $\{p_0, q(\lambda)\} = q(\lambda)$ and $\{p(\mu), q_0\} = q(\mu)$, we obtain

$$\{q(\lambda), p(\mu)\}' = \frac{q(\lambda)p(\mu) - q(\mu)p(\lambda)}{\lambda - \mu} + \frac{q(\mu)q(\lambda)}{q_0}.$$

Finally, we have

$$\{w(\lambda), w(\mu)\}' = (w(\lambda) - w(\mu))\left(\frac{w(\lambda) - w(\mu)}{\lambda - \mu} - \frac{w(\lambda)w(\mu)}{q_0}\right).$$

This becomes (6.5) for a particular choice $\Phi_2 = \log q_0 = 0$. □

8. The canonical coordinates on Rat'_N

Now we turn to the submanifold Rat'_N with $q_0 = \sum \rho_k = 1$ and the Poisson bracket (6.5). Here the situation is a little more subtle. In all formulae we omit the prime near the bracket $\{\bullet, \bullet\}'$.

Theorem 4. *The bracket (6.5) in λ–ρ coordinates has the form*

$$\{\rho_k, \rho_n\} = \left[\frac{2\rho_k\rho_n}{\lambda_n - \lambda_k} - 2\rho_k\rho_n\left(\sum_{s\neq k}\frac{\rho_s}{\lambda_s - \lambda_k} - \sum_{s\neq n}\frac{\rho_s}{\lambda_s - \lambda_n}\right)\right](1 - \delta_k^n), \tag{8.1}$$

$$\{\rho_k, \lambda_n\} = -\rho_k\rho_n + \rho_k\delta_k^n, \tag{8.2}$$

$$\{\lambda_k, \lambda_n\} = 0. \tag{8.3}$$

Proof. The proof is similar to the proof of Theorem 2 and therefore is omitted. □

The theorem implies

$$\{q(\lambda_k), \lambda_n\} = -q(\lambda_k)\rho_n + q(\lambda_k)\delta_k^n. \tag{8.4}$$

Thus, we have the first system of canonical coordinates, the so-called the action–angle variables

$$\lambda_1, \ldots, \lambda_{N-1}, \qquad \theta_1, \ldots, \theta_{N-1}, \tag{8.5}$$

where θ's defined by (5.3) are real and canonically paired with λ's. Indeed, (8.4) implies

$$\{\theta_k, \lambda_n\} = (\delta_k^n - \delta_0^n).$$

Formulae (8.1) and (8.4) produce

$$\{\log q(\lambda_k), \log q(\lambda_n)\} = \sum_{s\neq k} \frac{\rho_s+\rho_k}{\lambda_k-\lambda_s} - \sum_{s\neq n} \frac{\rho_s+\rho_n}{\lambda_n-\lambda_s}. \tag{8.6}$$

This identity implies the commutativity of the angles: $\{\theta_k, \theta_n\} = 0$.

Using the theorem, it can be checked easily that $p_0 = -\sum \lambda_k$ is a Casimir of the bracket (6.5).

Evidently, for the restricted bracket (6.5) the canonical relations established in Theorem 3 survive. We have the second set of canonical variables on Rat'_N

$$\gamma_1, \dots, \gamma_{N-1}, \qquad \pi_1, \dots, \pi_{N-1}, \qquad \pi_k = \log(-1)^{N+k} p(\gamma_k). \tag{8.7}$$

These divisor-quasimomentum coordinates were introduced in [18]. The denominator $p(\lambda) = (-1)^N \prod(\lambda_n - \lambda)$ satisfies

$$(-1)^{N+k} p(\gamma_k) > 0, \quad k = 1, \dots, N-1.$$

Whence π's are real and canonically paired with γ's

$$\{\pi_n, \gamma_k\} = \delta_n^k.$$

All other brackets vanish.

In the rest of this section we show that the variables (5.5) associated with representation of $w(\lambda) \in \mathrm{Rat}'_N$ in the exponential form (3.2)

$$w(\lambda) = -\frac{1}{\lambda}\, \mathrm{e}^{\Xi(\lambda)}$$

can be moved by corresponding λ's

$$\{\theta'_k, \lambda_n\} = \delta_n^k, \quad k, n = 1, \dots, N-1.$$

Though it follows from the previous discussion of the action–angle variables and formula (5.6) we will give an independent proof of this fact. It is important to notice that we cannot expect commutativity of the variables θ'_n.

The multi-valued function $\Xi(\lambda)$ has the form $\Xi(\lambda) = \sum_{s=1}^{N-1} \log(\gamma_s - \lambda) - \log(\lambda_s - \lambda)$ and defined up to an integer multiple of $2\pi \mathrm{i}$. The bracket $\{\bullet, \Xi(\lambda)\}$ is single valued since additive constant vanishes. The Poisson bracket (6.5) in terms of the function $\Xi(\lambda)$ has the form

$$\{\Xi(\lambda), \Xi(\mu)\} = \frac{4\sinh^2(\Xi(\lambda) - \Xi(\mu) - \log\lambda + \log\mu/2)}{\lambda-\mu} + \frac{1}{\lambda}\,\mathrm{e}^{\Xi(\lambda)} - \frac{1}{\mu}\,\mathrm{e}^{\Xi(\lambda)}$$

or

$$\{\Xi(\lambda), \Xi(\mu)\} = \frac{1}{w(\lambda)w(\mu)} \frac{(w(\lambda)-w(\mu))^2}{\lambda-\mu} - w(\lambda) + w(\mu), \tag{8.8}$$

which is more convenient for calculations. The pole λ_k can be represented as a contour integral

$$\lambda_k = -\frac{1}{2\pi \mathrm{i}} \int_{O_k} \zeta\, \mathrm{d}\Xi(\zeta) = -\lambda'_k + \frac{1}{2\pi \mathrm{i}} \int_{O_k} \Xi(\zeta)\, \mathrm{d}\zeta,$$

where λ_k' is an arbitrary fixed point on the contour O_k surrounding λ_k. As a simple example we prove commutativity of λ's

$$\{\lambda_k, \lambda_n\} = \frac{1}{(2\pi \mathrm{i})^2} \int_{O_k} \int_{O_n} \{\Xi(\zeta), \Xi(\eta)\}\, \mathrm{d}\zeta\, \mathrm{d}\eta.$$

From formula (8.8) one can easily see that the double integral vanishes.

Now for the angles θ_n' we have

$$\{\theta_k', \lambda_n\} = \lim_{\lambda \to \lambda_k} \left[\{\Xi(\lambda), \lambda_n\} - \{\Xi(0), \lambda_n\} + \left\{ \log \lambda_k - \frac{\lambda}{\lambda_k}, \lambda_n \right\} \right].$$

The last term vanishes. The first two terms are more complicated

$$\{\Xi(\lambda), \lambda_n\} = \frac{1}{2\pi \mathrm{i}} \int_{O_n} \{\Xi(\lambda), \Xi(\zeta)\}\, \mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \int_{O_n} \frac{w(\lambda)}{w(\zeta)} \frac{\mathrm{d}\zeta}{\lambda - \zeta}, \tag{8.9}$$

$$-\frac{2}{2\pi \mathrm{i}} \int_{O_n} \frac{\mathrm{d}\zeta}{\lambda - \zeta}, \tag{8.10}$$

$$+\frac{1}{2\pi \mathrm{i}} \int_{O_n} \frac{w(\zeta)}{w(\lambda)} \frac{\mathrm{d}\zeta}{\lambda - \zeta}, \tag{8.11}$$

$$-\frac{1}{2\pi \mathrm{i}} \int_{O_n} [w(\lambda) - w(\zeta)]\, \mathrm{d}\zeta. \tag{8.12}$$

If $\lambda \to \lambda_k$, $k \neq n$ or $\lambda \to \lambda_0 = 0$, the terms (8.9)–(8.11) vanish. The term (8.12) is equal to $-\rho_n$. Therefore,

$$\lim_{\lambda \to \lambda_k} \{\Xi(\lambda), \lambda_n\} = -\rho_n, \quad k \neq n, \quad \{\Xi(0), \lambda_n\} = -\rho_n.$$

This implies $\{\theta_k', \lambda_n\} = 0$, $k \neq n$.

Furthermore, if $\lambda \to \lambda_n$, then the term (8.9) becomes -1, the term (8.10) becomes 2 and (8.11) vanishes. The term (8.12) is $-\rho_n$. Therefore,

$$\lim_{\lambda \to \lambda_n} \{\Xi(\lambda), \lambda_n\} = 1 - \rho_n.$$

Thus $\{\theta_n', \lambda_n\} = 1$.

9. Tangent and transversal flows

Using the poles of $w(\lambda)$ we define Hamiltonians $H_j = (1/j) \sum \lambda_n^j$, $j = 1, \ldots, N$. The flows produced by them in the bracket (6.5) are tangent to the isospectral manifold: $\{\lambda_k, H_j\} = 0$. Due to Theorem 4 the standard [21] Toda flows have the form

$$\rho_k^{\bullet} = \{\rho_k, H_j\} = \left(\lambda_k^{j-1} - \sum \lambda_n^{j-1} \rho_n \right) \rho_k, \quad k = 0, \ldots, N-1.$$

Toda flows commute $\{H_j, H_k\} = 0$ and linearized in the variables (8.5)

$$\theta_k^{\bullet} = \{\theta_k, H_j\} = (\lambda_k^{j-1} - \lambda_0^{j-1}), \quad k = 1, \ldots, N-1.$$

Similarly, from zeros of $w(\lambda)$ we define another set of Hamiltonians $T_j = (1/j)\sum \gamma_n^j$, $j = 1, \ldots, N-1$. By Theorem 3 the flows produced by these Hamiltonians do not affect γ's. Therefore, we call these commuting flows transversal. They are linearized in the variables (8.7)

$$\pi_k^{\bullet} = \{\pi_k, T_j\} = \gamma_k^{j-1}, \quad k = 1, \ldots, N-1.$$

This is an example of the situation similar to the one considered in physics [6,11]. Given two systems of canonical coordinates and two families of commuting Hamiltonians. Each family depends only on the half of the coordinates of the corresponding canonical system. Hamiltonians of both families produce coordinate system for the Poisson manifold.

It is routine exercise to derive the equations of motion for γ's under H flows and for λ's under T flows. Then the inverse spectral problem can be solved using trace formulae of Section 2. We do not dwell on this.

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